Abstract

The objectives of this work are to understand why non-trivial roots, $\rho = \beta + \mathrm{i}\cdot\gamma$, occur in the Riemann zeta function, $\zeta(s) = \zeta(\sigma + \mathrm{i}\cdot t)$, to define these roots mathematically, and to resolve the Riemann hypothesis.

At the roots of the Riemann zeta function, the two infinite Dirichlet series parts are coincidently divergent and are geometrically identical. The roots of the zeta function are the only points in the critical strip where infinite summation and infinite integration of the Dirichlet series terms are geometrically equivalent. Similarly, the roots of the zeta function with the real part of the argument reflected across the critical strip are the only points where infinite summation and infinite integration of the series parts with reflected argument, $1-s$, are geometrically equivalent.

Roots occur in the Riemann zeta function in the critical strip if and only if:

$$\lim_{m\to\infty}\left\{\left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(2m-1)\pi}{2t}\right]\right\}}\frac{\cos[t\cdot\log_e(k)]}{k^{\sigma}}\right]-\int_1^{\exp\left[\frac{(2m-1)\pi}{2t}\right]}\frac{\cos[t\cdot\log_e(x)]}{x^{\sigma}}dx\right\}\to 0$$

$$\lim_{m\to\infty}\left\{\left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(m-1)\pi}{t}\right]\right\}}\frac{\sin[t\cdot\log_e(k)]}{k^{\sigma}}\right]-\int_1^{\exp\left[\frac{(m-1)\pi}{t}\right]}\frac{\sin[t\cdot\log_e(x)]}{x^{\sigma}}dx\right\}\to 0$$

and

$$\lim_{m\to\infty}\left\{\left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(2m-1)\pi}{2t}\right]\right\}}\frac{\cos[t\cdot\log_e(k)]}{k^{1-\sigma}}\right]-\int_1^{\exp\left[\frac{(2m-1)\pi}{2t}\right]}\frac{\cos[t\cdot\log_e(x)]}{x^{1-\sigma}}dx\right\}\to 0$$

$$\lim_{m\to\infty}\left\{\left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(m-1)\pi}{t}\right]\right\}}\frac{\sin[t\cdot\log_e(k)]}{k^{1-\sigma}}\right]-\int_1^{\exp\left[\frac{(m-1)\pi}{t}\right]}\frac{\sin[t\cdot\log_e(x)]}{x^{1-\sigma}}dx\right\}\to 0$$

Reduced asymptotic expansions for the series parts at the roots of the zeta function, equated algebraically with reduced asymptotic expansions for the series terms with reflected argument at the roots of the zeta function with reflected argument, constrain the values of the real parts of both arguments to the critical line. Hence, the Riemann hypothesis is true.

At the roots of the zeta function, the real part of the argument is the exponent, and the real and imaginary parts together constitute the coefficient of proportionality in a geometrical constraint of the discrete partial sums of the series parts by a continuous and divergent

envelope. The imaginary part of the roots of the Riemann zeta function, γ, is defined by simultaneous solutions of the following two formulae:

$$\lim_{m \to \infty} \left\{ \left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(2m-1)\pi}{2\gamma}\right]\right\}} \frac{\cos[\gamma \cdot \log_e(k)]}{k^{1/2}} \right] + (-1)^m \cdot \frac{\gamma}{\left[\left(\frac{1}{2}\right)^2 + \gamma^2\right]} \cdot \exp\left[\frac{(2m-1)\pi}{2\gamma}\right]^{1/2} \right\} \to 0$$

$$\lim_{m \to \infty} \left\{ \left[\sum_{k=1}^{floor\left\{\exp\left[\frac{(m-1)\pi}{\gamma}\right]\right\}} \frac{\sin[\gamma \cdot \log_e(k)]}{k^{1/2}} \right] - (-1)^m \cdot \frac{\gamma}{\left[\left(\frac{1}{2}\right)^2 + \gamma^2\right]} \cdot \exp\left[\frac{(m-1)\pi}{\gamma}\right]^{1/2} \right\} \to 0$$

$$m = 1, 2, 3, \ldots$$

Values of the imaginary part of the first 50 roots of the Riemann zeta function are calculated to 80 correct significant figures using a notebook computer by numerically solving the formulae above with proportionate, finite values of m. The first five solutions are:

γ=14.134725141734693790457251983562470270784257115699243175685567460149963429809256...
γ=21.022039638771554992628479593896902777334340524902781754629520403587598586068890...
γ=25.010857580145688763213790992562821818659549672557996672496542006745092098441644...
γ=30.424876125859513210311897530584091320181560023715440180962146036993329389333277...
γ=32.935061587739189690662368964074903488812715603517039009280003440784815608630551...

It is also demonstrated using the notebook computer that the two formulae above yield calculated values of the imaginary part of the roots of the Riemann zeta function with more than 330 correct significant figures.